# Optical properties of semiinfinite turbid media: some simple analytical approximations

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**Abstract.** This work is devoted to the derivation of simple approximate equations, which can be used for studies of light reflection from various semiinfinite weakly absorbing turbid media. It is assumed that the probability of photon absorption in a single scattering event is small. The method used is largely based on the approximate summation of the MacLauren series for total light reflectance and other radiative transfer characteristics as functions of the probability of photon absorption. The main results comprise convenient analytical expressions for total light reflectance, plane albedo, reflection function, and degree of polarization of reflected light. 
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## 1 Introduction

The core of the radiative transfer theory is the vector integrodifferential radiative transfer equation,[1] which describes the intensity and polarization of diffused waves in scattering media. Generally speaking, most of polarization and transfer characteristics of various media can be derived using this equation. However, the equation is not very convenient for analytical studies. Thus, there is a need for the derivation of simple analytical equations, which can describe the transport of light in random media.

A number of analytical approximations for various light field characteristics are discussed by van de Hulst in his fundamental monograph on the subject of radiative transfer.[2] They are largely based on various transformations and approximations of the radiative transfer equation.

It is of interest, however, to derive analytical equations, well known in radiative transfer theory, but actually not using the radiative transfer equation explicitly. These approximate equations can be used in numerous applications of the radiative transfer theory, especially as far as optical engineering is concerned. It could be of interest also from a didactic point of view.

We consider here only one selected problem of the radiative transfer theory, namely the reflection of light from semiinfinite media (e.g., snow fields, paints, whitecaps, etc.) However, other more complex cases, including finite scattering layers, can be handled using the same chain of arguments.

The reflection of light from a semiinfinite turbid medium is a standard problem of radiative transfer.[1] Actually optical properties of finite but optically thick slabs can easily be found if the reflection function of a semiinfinite layer with the same local optical characteristics is known.[2,3] This is why that problem was studied in great detail both theoretically and experimentally. Especially fruitful results as far as reflectance spectroscopy is concerned were obtained by Rozenberg.[4–6] They were updated and brought to correspondence with the exact asymptotic radiative transfer theory by Zege, Ivanov, and Katsev.[7]

Our primary task is to present a new method for the derivation of analytical equations for total reflectance and other reflection characteristics of semiinfinite turbid media. Note that the method proposed can be used well beyond the narrow scope considered here.

## 2 Total Reflectance of a Semiinfinite Turbid Layer

Let us start from the derivation of an approximate formula for the total reflectance $r_s$ of a semiinfinite layer. The value of $r_s$ is defined as follows[1]:

$$r_s=\frac{1}{2\pi}\int_0^{2\pi}d\psi\int_0^{\pi/2}\sin 2\vartheta d\vartheta \times\int_0^{\pi/2}\sin 2\vartheta_0 d\vartheta_0 R(\vartheta_0,\vartheta,\psi),$$

where $R(\vartheta_0,\vartheta,\psi)$ is the reflection function[2] of a plane-parallel turbid layer, $\vartheta_0$ is the incidence angle, $\vartheta$ is the observation angle, and $\psi$ is the relative azimuth between incident and reflected light beams. The reflection function $R(\vartheta_0,\vartheta,\psi)$ is defined as the ratio of the intensity of reflected light for a given scattering medium to that of an absolutely white Lambertian surface. By definition, we have for such a surface: $R\equiv 1$ and, therefore, $r_s=1$.

First of all, we note that the value of $r_s(\omega_0)$ for semiinfinite turbid media can be presented in the following general form:

$$r_s(\omega_0)=\sum_{n=1}^{\infty}a_n\omega_0^n, \tag{1}$$

where $a_n$ are unknown coefficients and $\omega_0$ is the probability of photon survival in a single light scattering event (so-called single scattering albedo[1]). We assume that $\omega_0$ is close to 1, which is the case for many substances in visible and near-infrared regions of the electromagnetic spectrum.[5] Then it is more convenient to use the expansion:

$$r_s(\beta)=\sum_{n=1}^{\infty} a_n(1-\beta)^n, \tag{2}$$

where $\beta\equiv 1-\omega_0$ is the probability of photon absorption, which is assumed to be a small number. Clearly, we have from Eq. (2) at $\beta=0$:

$$r_s(0)=\sum_{n=1}^{\infty} a_n\,. \tag{3}$$

However, it also follows:

$$r_s(0)=1, \tag{4}$$

which is due to the energy conservation law. Thus, one obtains that

$$\sum_{n=1}^{\infty} a_n=1, \tag{5}$$

and numbers $a_n$ can be interpreted in terms of the probability theory. In particular, the value of $a_1$ gives us the probability that the photon will be singly scattered before escaping the turbid medium. The probabilities of scattering events $a_1$, $a_2$, $a_3$,... do not depend on each other. The theorem of adding independent probabilities brings us to Eq. (5) as well. Let us substitute the following exact expansion in Eq. (2):

$$(1-\beta)^n=\sum_{j=0}^{n} (-1)^j \binom{n}{j}\beta^j, \tag{6}$$

where

$$\binom{n}{j}\equiv\frac{n!}{j!(n-j)!}. \tag{7}$$

Then it follows from Eqs. (2) and (6):

$$r_s(\beta)=\sum_{n=1}^{\infty} a_n\sum_{j=0}^{n} (-1)^j \binom{n}{j}\beta^j, \tag{8}$$

or in the explicit form:

$$r_s(\beta)=\sum_{n=1}^{\infty} a_n\left[1-\beta n+\frac{\beta^2 n(n-1)}{2} - \frac{\beta^3 n(n-1)(n-2)}{6}+\ldots\right]. \tag{9}$$

Equation (9) can be rewritten in the following form:

$$r_s=1-\beta\langle n\rangle+\frac{\beta^2\langle n(n-1)\rangle}{2}-\frac{\beta^3\langle n(n-1)(n-2)\rangle}{6}+\ldots, \tag{10}$$

where we used the normalization condition in Eq. (5) and defined the following averages:

$$\langle n\rangle=\sum_{n=1}^{\infty} na_n\,,\quad \langle n(n-1)\rangle=\sum_{n=1}^{\infty} n(n-1)a_n\,,$$

$$\langle n(n-1)(n-2)\rangle=\sum_{n=1}^{\infty} n(n-1)(n-2)a_n\,, \tag{11}$$

and so on. Here $\langle n\rangle$ is the average number of scattering events in the medium.

Equation (10) is an exact formula. Now we make some assumptions to simplify Eq. (10). First, we assume that $\langle n\rangle$ is large and, consequently, $\langle n(n-1)\rangle\approx\langle n^2\rangle$, $\langle n(n-1)(n-2)\rangle\approx\langle n^3\rangle$ and so on, are also large. Clearly, such an approximation is valid for weakly absorbing media (as $\beta\to 0$). Note that for nonabsorbing semiinfinite media $\langle n\rangle\to\infty$. This gives us, instead of Eq. (8):

$$r_s=1-\beta\langle n\rangle+\frac{\beta^2}{2}\langle n^2\rangle-\frac{\beta^3}{6}\langle n^3\rangle+\ldots \tag{12}$$

or

$$r_s=\langle \exp(-\beta n)\rangle, \tag{13}$$

where we used the expansion

$$\exp(-\beta n)=\sum_{k=0}^{\infty}\frac{(-1)^k(\beta n)^k}{k!}. \tag{14}$$

Thus, the value of $r_s$ is given by

$$r_s=\sum_{n=1}^{\infty} \exp(-\beta n)a_n\,. \tag{15}$$

Applying the approximate formula,

$$\sum_{n=1}^{\infty} f(n)=\int_0^{\infty} f(x)dx, \tag{16}$$

we have:

$$r_s=\int_0^{\infty} \exp(-\beta x)a(x)dx, \tag{17}$$

or using the mean value theorem:

$$r_s=\exp[-\beta\overline{x(\beta)}]. \tag{18}$$

We also used the integral form of the normalization condition in Eq. (5):

$$\int_0^\infty a(x)dx=1. \quad (19)$$

Note that we did not specify any specific laws of scattering probability in the derivation of Eq. (10). Thus, Eq. (18) can be applied in a much broader context than just scattering of photons by particles in a turbid medium. Comparison of Eqs. (12) and (18) shows us that $\overline{x}\rightarrow\langle n\rangle$ as $\beta\rightarrow 0$. However, generally speaking, $\overline{x}\neq\langle n\rangle$. This is due to the fact that differences $\sigma_j=\langle n^j\rangle-\langle n\rangle^j$ $(j=2,3...\infty)$ are nonzero in general.

The problem we face now is the determination of the parameter $\overline{x}$. For this we use the well-known asymptotic result of the radiative transfer theory[1,2]:

$$r_s=1-4\left[\frac{\beta}{3(1-g)}\right]^{1/2}, \quad (20)$$

which is valid as $\beta\rightarrow 0$. Here

$$g=\frac{1}{2}\int_0^\pi \cos\theta p(\theta)\sin\theta d\theta, \quad (21)$$

and $p(\theta)$ is the phase function, which describes the probability of photon scattering in a particular direction, specified by the scattering angle $\theta$. It is normalized as follows:

$$\frac{1}{2}\int_0^\pi p(\theta)\sin\theta d\theta=1. \quad (22)$$

Clearly, for equal scattering probabilities: $p(\theta)=1$ and $g=0$. For phase functions highly elongated in the forward direction, which is the case for particles in atmosphere and ocean, $g$ is close to 1.[2]

Equation (18) takes the following form as $\beta\rightarrow 0$:

$$r_s=1-\beta\overline{x}. \quad (23)$$

So, comparing Eqs. (20) and (23) we have

$$\overline{x}=\frac{4}{k}, \quad (24)$$

where

$$k=[3(1-g)\beta]^{1/2} \quad (25)$$

is the diffusion factor of the radiative transfer theory. Note that in the diffusion approximation we have, instead of Eq. (25)[8]:

$$k_d=[3(1-g\omega_0)\beta]^{1/2}.$$

The difference between $k$ and $k_d$ is small in this case, which we consider here $(\omega_0\rightarrow 1)$.

Equation (24) shows that the average number of scattering events in a semiinfinite layer is proportional to $k^{-1}$ as $\beta\rightarrow 0$. In particular, we have: $\overline{x}\rightarrow\infty$ as $\beta\rightarrow 0$ or $g\rightarrow 1$.

Finally, we have from Eqs. (18) and (24):

$$r_s=\exp\left(-\frac{4\beta}{k}\right). \quad (26)$$

Equation (26) presents the total reflectance $r_s$ of a semiinfinite weakly absorbing medium as a function of two parameters. They are the probability of photon absorption $\beta$ and the diffusion factor $k$.

The value of $k$ determines the decrease of photon flux in deep layers of absorbing semiinfinite layers. Namely, it follows for the flux $F$ at the depth $z$[2]:

$$F=F_0\exp\left(-\frac{z}{L_d}\right), \quad (27)$$

where $F_0$ is the incident light flux and

$$L_d=(k\sigma_{\text{ext}})^{-1} \quad (28)$$

is the so-called diffusion length. Note that Eq. (27) cannot be applied for nonabsorbing media.[2] The value of $\sigma_{\text{ext}}$ gives us the extinction coefficient. Introducing the absorption length $L_{\text{abs}}$, which is inversely proportional to the absorption coefficient $\sigma_{\text{abs}}$,

$$L_{\text{abs}}=(\sigma_{\text{abs}})^{-1}, \quad (29)$$

we have from Eq. (26):

$$r_s=\exp\left(-\frac{4L_d}{L_{\text{abs}}}\right). \quad (30)$$

Thus, the total reflectance is determined by the ratio of the diffusion length $L_d$ to the absorption length $L_{\text{abs}}$:

$$\xi=\frac{L_d}{L_{\text{abs}}}. \quad (31)$$

For small values of $\xi$, the value of $r_s$ is close to 1 and we have approximately:

$$\xi=\frac{1-r_s}{4}. \quad (32)$$

This value could be easily found experimentally for a given turbid medium.

Here we come to the important conclusion. Namely, Eq. (26) can be applied to a much broader class of scattering processes than those described by the radiative transfer equation. To make things more clear, let us consider foamed media (e.g., whitecaps over ocean). The notion of the phase function can be hardly introduced for such a densely packed system of air bubbles. However, both the value $L_{\text{abs}}$ and $L_d$ can be defined and in principle measured [see Eqs. (27) and (30)]. The application of Eq. (26) to the interpretation of light reflectance measurements for snow and foam samples has shown a high accuracy of the theoretical model proposed.[8] This is mostly due to the fact that the photon free path length both in snow and foam is much

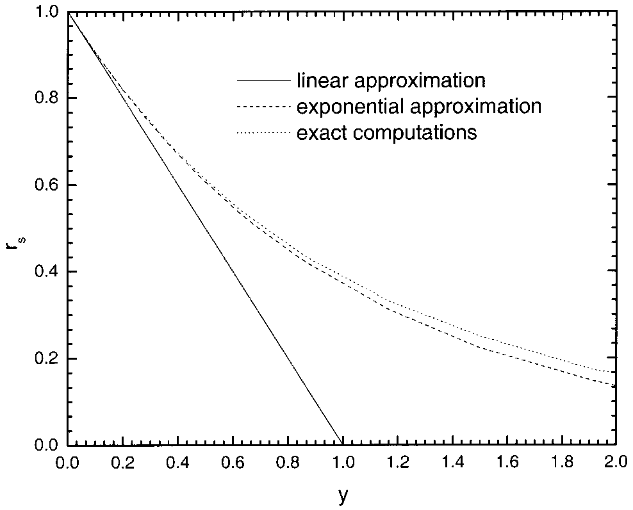


**Fig. 1** The dependence $r_s(y)$, calculated with Eqs. (20) (linear approximation) and (33) (exponential approximation) and the exact radiative transfer code.[9]

larger than the wavelength. Then the conventional radiative transfer equation applies (with modified parameters of an elementary volume).[8]

Let us rewrite Eq. (26) in the more explicit form:

$$r_s=\exp\left(-4\left[\frac{1-\omega_0}{3(1-g)}\right]^{1/2}\right). \tag{33}$$

This equation is often used in the applied radiative transfer theory.[7] It can be applied to a much wider class of absorbing media than the simple linear approximation of Eq. (20). We notice that the combination of local optical characteristics, given by

$$y=4\left[\frac{1-\omega_0}{3(1-g)}\right]^{1/2} \tag{34}$$

is of paramount importance in the radiative transfer. It determines the total reflectance from a weakly absorbing semiinfinite layer.

The accuracy of Eqs. (20) and (33) as compared to calculations with the numerical solution of the radiative transfer equation for water clouds with different radii of particles is presented in Fig. 1. Specifically, we used the case of water droplets size distribution, given by $f(a)=Na^6\exp(-9a/a_{\mathrm{ef}})$, where the effective radius $a_{\mathrm{ef}}$ was assumed to be equal to 4, 6, and 16 $\mu$m. Here $N=9^7/[\Gamma(7)a_{\mathrm{ef}}^7]$ and $\Gamma(7)$ is the gamma function. Calculations were performed at a wavelength of 1.55 $\mu$m. The refractive index of droplets was assumed to be equal to $1.3109-\chi i$, where $\chi$ was varied in the range 0.00001 to 0.001. The values of $g$ and $\omega_0$ were found using the Mie theory. Exact results for all $\chi$ and $a_{\mathrm{ef}}$ used in calculations lay on the same curve plotted against $y$. This is a well-known feature of radiative transfer in semiinfinite media, having the same value of $y$.[2]

We see that the accuracy of Eq. (33) is much better than that of an exact asymptotic result as in Eq. (20). Also it follows that Eq. (33) can be used at least up to $y=1/2$ regardless actual values of $g$ and $\omega_0$. It means that the smaller the value of $g$, the larger the applicability of Eq. (33) in terms of the probability of photon absorption $\beta$. In particular, we obtain from Eq. (34) at $y=1/2$:

$$\beta_{1/2}=\frac{3}{64}(1-g). \tag{35}$$

Here $\beta_{1/2}$ is the value of $\beta$, correspondent to $y=1/2$ at a fixed $g$. We see that the value of $\beta_{1/2}=3/64$ (or approximately 1.6%) at $g=0$. It is much smaller for values of $g$, which are close to 1. For instance, $\beta_{1/2}$ is only 0.07% at $g=0.85$, which is typical for water clouds. Note that Eq. (33) can be applied for larger $y$ (e.g., $y\in[1/2;1]$, see Fig. 1), if a high accuracy is not a primary requirement (e.g., for rapid estimations).

Our derivation underlines assumptions, which lead to the approximation of Eq. (33). Note that Eq. (18) follows directly from Eq. (13), assuming that $\sigma_j=\langle n^j\rangle-\langle n\rangle^{j\to 0}$ for all $j$ and $\overline{x}=\langle n\rangle$. Then,

$$\langle\exp(-\beta n)\rangle=\exp(-\beta\langle n\rangle). \tag{36}$$

This is in fact a major assumption used in the derivation of Eq. (33).

## 3 Plane Albedo and the Reflection Function

The same line of reasoning can be applied to the derivation of the solution for the plane albedo $r_p(\vartheta_0)$, defined as

$$r_p(\vartheta_0)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\int_0^{\pi/2}\sin 2\vartheta d\vartheta R(\vartheta_0,\vartheta,\psi).$$

Again, we have for an absolutely white Lambertian surface: $R\equiv 1$ and, therefore, $r_p=1$ for all incidence angles $\vartheta_0$.

The MacLauren expansion of $r_p(\vartheta_0)$ with respect to $\omega_0$ has the following form:

$$r_p(\vartheta_0)=\sum_{n=1}^{\infty} b_n(\vartheta_0)\omega_0^n, \tag{37}$$

where unknown coefficients $b_n$ depend now on the incidence angle $\vartheta_0$. Equation (20) takes the following form for the plane albedo[2]:

$$r_p(\vartheta_0)=1-u_0(\vartheta_0)y, \tag{38}$$

where $y$ is given by Eq. (34) and $u_0(\vartheta_0)$ is a so-called escape function[2] of the radiative transfer theory. It can be approximated by the following formula outside the region of the grazing incidence angles $\vartheta_0$[1]:

$$u_0(\vartheta_0)=\frac{3}{7}(1+2\cos\vartheta_0). \tag{39}$$

Equation (39) describes also the angular variation of light transmitted by overcast clouds. Then $\vartheta_0$ in Eq. (39) should be substituted by $\vartheta$.[2]

Following the same steps as before, it follows that, instead of Eq. (18):

$$r_p(\vartheta_0)=\exp[-\beta\overline{x(\vartheta_0)}], \tag{40}$$

where

$$\overline{x(\vartheta_0)}=\frac{4u_0(\vartheta_0)}{k}. \tag{41}$$

Also we can write:

$$r_p(\vartheta_0)=\exp[-yu_0(\vartheta_0)]. \tag{42}$$

It follows from Eqs. (39) and (41) that the average number of scattering events $\overline{x(\vartheta_0)}$ as $\beta\rightarrow 0$ is larger for smaller angles $\vartheta_0$. This effect follows for physical grounds as well. Indeed, it is harder for photons to escape the medium if they are injected along the normal to the scattering medium. We see, therefore, that the contribution of multiple scattering decreases with $\vartheta_0$.

The final task is to obtain the similar approximate equation for the reflection function of a semiinfinite medium $R_\infty(\vartheta_0,\vartheta,\psi)$.[2] We use the following MacLauren expansion in this case:

$$R_\infty(\vartheta_0,\vartheta,\psi)=\sum_{n=1}^{\infty} c_n(\vartheta_0,\vartheta,\psi)\omega_0^n. \tag{43}$$

$R_\infty(\vartheta_0,\vartheta,\varphi)$ is not necessarily equal to one at $\omega_0=1$. So, we have:

$$R_\infty^0(\vartheta_0,\vartheta,\psi)\equiv\sum_{n=1}^{\infty} c_n(\vartheta_0,\vartheta,\psi)\neq 1. \tag{44}$$

The subscript 0 means that the quantity $R_\infty^0$ corresponds to the case of zero absorption $[R_\infty^0\equiv R_\infty(\omega_0=1)]$. Let us consider the ratio

$$\Upsilon=\frac{R_\infty(\vartheta_0,\vartheta,\psi)}{R_\infty^0(\vartheta_0,\vartheta,\psi)}. \tag{45}$$

It can be represented by the following expansion:

$$\Upsilon=\sum_{n=1}^{\infty} d_n(\vartheta_0,\vartheta,\psi)\omega_0^n, \tag{46}$$

where

$$d_n(\vartheta_0,\vartheta,\psi)=\frac{c_n(\vartheta_0,\vartheta,\psi)}{\sum_{n=1}^{\infty}c_n(\vartheta_0,\vartheta,\psi)}. \tag{47}$$

It follows that

$$\sum_{n=1}^{\infty} d_n(\vartheta_0,\vartheta,\psi)=1, \tag{48}$$

and numbers $d_n$ can be interpreted in terms of probabilities.

Repeating the previous steps, we arrive at the following formula:

$$\Upsilon=\exp[-\beta\overline{x(\vartheta_0,\vartheta,\psi)}]. \tag{49}$$

Again we use the exact asymptotic radiative transfer theory to get the average scatterings number $\overline{x(\vartheta_0,\vartheta,\psi)}$. Namely, it follows that[2]:

$$\Upsilon=1-h(\vartheta_0,\vartheta,\psi)y, \tag{50}$$

where

$$h(\vartheta_0,\vartheta,\psi)=\frac{u_0(\vartheta_0)u_0(\vartheta)}{R_\infty^0(\vartheta_0,\vartheta,\psi)}. \tag{51}$$

On the other hand, we have from Eq. (49):

$$\Upsilon=1-\beta\overline{x(\vartheta_0,\vartheta,\psi)}. \tag{52}$$

Therefore, comparing Eqs. (50) and (52), we arrive to the following important equation:

$$\overline{x(\vartheta_0,\vartheta,\psi)}=\frac{4h(\vartheta_0,\vartheta,\psi)}{k}. \tag{53}$$

Finally, it follows [see Eqs. (45) and (49)] that:

$$R_\infty(\vartheta_0,\vartheta,\psi)=R_\infty^0(\vartheta_0,\vartheta,\psi)\exp[-yh(\vartheta_0,\vartheta,\psi)], \tag{54}$$

where we used the equality: $y=4\beta/k$.

Formulas (33), (40), and (54) have been known for a long time.[7,8] However, the new way of deriving them can facilitate applications. Also we note that the function $h$, given by Eq. (51), has a simple physical sense. It is equal to the ratio $\overline{x(\vartheta_0,\vartheta,\psi)}/\overline{x}$ [see Eqs. (53) and (24)]. Therefore, the function $h$ gives the ratio of the average number of scatterings for a given incidence, observation, and azimuth angles to that for the case of total reflectance.

Results presented previously can be generalized for the case of polarized light. Then the Stokes vector parameter with components $I$, $Q$, $U$, and $V$[2,8] should be used instead of diffused light intensity. In particular, let us consider the case of the normal illumination of a semiinfinite plane-parallel turbid layer. Then the degree of polarization of reflected light is given by:

$$P_\infty=-\frac{Q}{I}, \tag{55}$$

and components $U$ and $V$ vanish. The positive polarization means that oscillations of the electric vector are preferentially in the plane perpendicular to the plane containing the observing direction and the normal to the layer.[2,8]

It is known that the value of $I$ is much more sensitive to the change of the absorption in the medium than the parameter $Q$. To simplify, we assume that $Q$ in Eq. (55) coincides with its value for a nonabsorbing medium.

To study the validity range of this assumption, we have calculated values of $Q$ and $I$ using the doubling method of the vector radiative transfer equation solution, assuming

that the optical thickness $\tau$ is equal to 500, which is a close representation of a semiinfinite medium. The wavelength was equal to 0.65 $\mu$m. The phase function was kept constant and equal to that of water droplets with the same particle size distribution as in Fig. 1 (at $a_{ef}=6\ \mu m$). The value of the asymmetry parameter of $g$ was, therefore, constant and equal to 0.85. The single scattering albedo was changed in the range 0.95 to 1.0. This corresponds to the change of the probability of photon absorption $\beta$ from 0 until 5%. Then the parameter $y$ is changed in the range [0,4/3].

It should be stressed that such small variations of the single scattering albedo produce large variations of the value of $I$ and the reflection function[2]: $R=\pi I/\mu_0 E$. Here $E$ is the incident light flux density. We found, in particular, that $I$ changes more than four times at the nadir observation, and a 60-deg observation angle for changes of albedo in the range 0.95 to 1.0 (a semiinfinite medium case). The value of $Q$ changes only by 13% for the same conditions. The variation of $Q$ is below 5% for photon absorption probabilities below 2%.

Therefore, assuming that values of $Q$ for nonabsorbing and weakly absorbing media coincide, we obtain, using Eqs. (54) and (55):

$$P_\infty(\vartheta)=P_\infty^0(\vartheta)\exp[yh(0,\vartheta,0)], \tag{56}$$

where $P_\infty^0 \equiv -R_{\infty 21}/R_\infty^0$ and we used equalities

$$I=R_\infty I_0,\quad Q=R_{\infty 21} I_0, \tag{57}$$

which are valid in the case of incident unpolarized light having the intensity $I_0$. Here $R_{\infty 21}$ is the element 21 of the reflection matrix of a turbid medium.[8] We neglect the dependence of $R_{\infty 21}$ on $y$ and assume that $R_{\infty 21}=R_{\infty 21}^0$.

Equation (56) shows that the increase in absorption leads to an increase in light polarization, which is a well-established result.[10] Moreover, the ratio $P_\infty^0/P_\infty$ coincides with $\Upsilon$ [see Eqs. (49) and (56)].

Errors in Eqs. (54) and (56) are given in Fig. 2, assuming the same particle size distribution of water droplets as in Fig. 1 at $a_{ef}=6\ \mu m$, $\lambda=0.65\ \mu m$, $\vartheta_0=0$ deg, and $\vartheta=60$ deg. The single scattering albedo was changed in the range 0.95 to 1.0. We found that the error is smaller than 10% at $\omega_0>0.98$ ($R_\infty>0.32$) for $P_\infty$ [see Eq. (56)]. It is smaller than 8% at $\omega_0>0.95$ ($R_\infty>0.18$) for the value of $R_\infty$ [see Eq. (54)].

Therefore, approximations presented here can be used to estimate the degree of polarization and intensity of light reflected from weakly absorbing ($\omega_0\to 1$) semiinfinite ($\tau\to\infty$) media, avoiding the numerical solution of four coupled integrodifferential radiative transfer equations,[8] which could be of importance for optical engineering applications.

## 4 Conclusions

We derive a number of approximate equations, valid for the case of semiinfinite ($\tau\to\infty$) weakly absorbing ($\omega_0\to 1$) turbid media. They can be used in a wide range of applications, especially for the development of the spectroscopy and polarimetry of optically thick, weakly absorbing turbid media. One important limitation is the infinite thickness of the layer assumed, which is not always the case (e.g., for clouds). However, it should be pointed out that there are simple equations, which connect reflection and transmission characteristics of turbid layers with optical thicknesses larger than 5 with functions derived here [e.g., see Eq. (54)].[1–8,11–13] This extends the results presented here to a broader range of turbid media, including clouds, leaves, paper, and photolayers.

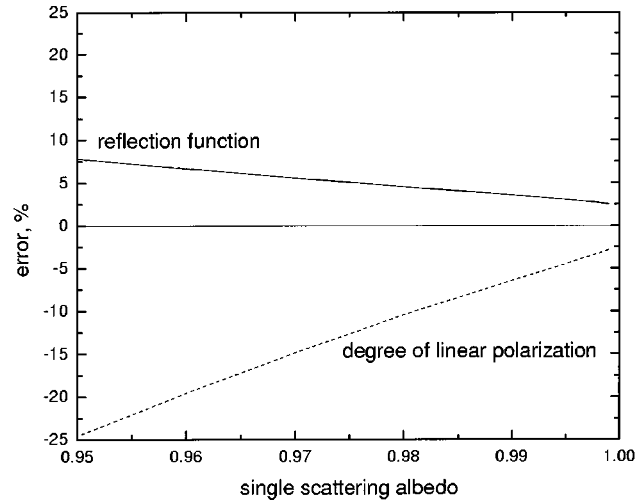


**Fig. 2** Errors of Eqs. (54) (solid line) and (56) (dashed line). Exact calculations were performed using the vector radiative transfer code for the same particle size distribution of water droplets as in Fig. 1 at $a_{ef}=6\ \mu m$, $\lambda=0.65\ \mu m$, $\vartheta_0=0$ deg, and $\vartheta=60$ deg.

Equations (33), (42), (54), and (56) can be used for the determination of the spectral dependence of a single scattering albedo of the scattering medium in question, which allows us in principle to find the spectral behavior of the imaginary part of the refractive index of scatterers. This can be used for spectroscopic purposes and the chemical composition of particles identification.[5]

It follows from Eqs. (56) that darker surfaces yield larger values of the degree of polarization. This is confirmed by experiments in Refs. 14 and 15.


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